

# Angular momentum quantization from Planck 's energy quantization

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We present in this work a pedagogical way of quantizing the atomic orbit for the hydrogen's atom model proposed by Bohr without using his hypothesis of angular momentum quantization. In contrast to the usual treatment for the orbital quantization, we show that using energy conservation, correspondence principle and Planck 's energy quantization Bohr's hypothesis can be deduced from and is a consequence of the Planck's energy quantization.

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## I. INTRODUCTION

The existence of an atomic nucleus was confirmed in 1911 by E. Rutherford in his classic scattering experiment [1]. Before that time, it was believed that an atom was something like a positively charged dough and negatively charged raisins scattered here and there within the dough, or an ice-cream with chocolate chip flakes in it, where the ice-cream would be the positive charge (protons) and the chocolate chip flakes the negative charge (electrons); an atomic model proposed by J.J.Thomson in 1904. However, this atomic model had many problems. And it became untenable as E. Rutherford showed the inconsistencies of such a model in face of his experiments of alpha particle (ionized helium) scattering by thin sheets of gold as targets. The main objection to that model being that scattering experiments indicated a far more "dilute" type of matter constituents and "empty" space within objects.

Then, Rutherford himself proposed the orbital model for the atom, in which there was a central nucleus populated by positively charged particles – protons – and negatively charge particles – electrons – moving around the nucleus in orbital trajectories, similar to the solar system with the sun at the center and planets orbiting it, described by the mechanics of celestial bodies acted on under central forces among them. Later on, the concept of neutral particles – neutrons – within the nucleus came to be added into the model.

The planetary model, however, was not free of problems. The main objection was that in such a model, electrons moving around the nucleus would radiate energy and therefore, classically, such an atom would collapse into itself after electrons radiated all their energy. Therefore, the model proposed by Rutherford had to be modified.

This modification came through the introduction of a completely radical concept unknown to classical physics, i.e., the quantization of physical quantities. In 1912 N. Bjerrum and a year later P. Ehrenfest were the first ones to apply successfully the quantum principles to the study of molecular spectroscopy. Following the suggestion of Lorentz and inspiring in the ideas of Nernst, they considered that a diatomic molecule could be seen as two atoms oscillating along the line that links the two, and that this line also rotates in a plan. This last rotational movement they supposed to be such that its energy would be multiple of  $hf$  (according to Bjerrum) or  $\frac{1}{2}hf$  (according to a later correction by Ehrenfest),  $f$  being the number of revolutions per second the molecule executed in this plan of rotation.

Two major changes in the building and understanding of the atomic models were developed at the beginning of XX century: firstly it was necessary to forgo the idea that the lines of the atomic spectra were due to the natural modes of atomic oscillations, and secondly that the spectral lines had to be seen as quantum phenomena or quantum effects.

In this line, Bohr was the first one who was able to explain the Balmer series of the atomic spectra by postulating the angular momentum quantization for the hydrogen atom 's orbitals.

We want to emphasize that in this hystoric development of the atomic model the quantization of the angular momentum was necessary to ensure the stability of the atomic orbital. In our work here, we start of imposing the stability condition for the atomic orbital and demonstrate that requiring the stable minimum for the Planck 's energy quantization, we arrive at the angular momentum quantization.

In the next section we present the Rutherford's orbital model as a general introduction. In section 3 we present Bohr's hypothesis for quantizing atomic orbitals. Section 4 is devoted to developing our main ideas concerning the requirement to minimize the Planck's energy and use it to identify with the Bohr's quantization hypothesis. Finally

the last section is for our conclusions.

## II. THE PLANETARY ATOMIC MODEL

The discoveries that occurred by the end of XIX century led the physicist Ernest Rutherford to do scattering experiments that culminated in a proposal for the planetary model for atoms.

According to this model, all positive charge of a given atom, with approximately 99% of its mass, would be concentrated in the atomic nucleus. Electrons would be moving around the nucleus in circular orbits and these would be the carriers of the negative charges. Knowing that the charge of an electron and the charge of a proton are the same in modulus, and that the nucleus has  $Z$  protons, we can define the charge of the nucleus as  $Ze^-$ .

Experimentally we observe that in an atom the distance  $r$  between the electron orbit and the nucleus is of the order  $10^{-10}m$ .

In this section, we build the planetary model for the atom and analyse its predictions compared to experimental data.

Using Coulomb's law

$$F = \frac{Z|e^-|^2}{4\pi\epsilon_0 r^2} \quad (1)$$

and the centripetal force acting on the electron in its circular orbit

$$F_c = \frac{m_e v^2}{r} \quad (2)$$

results in

$$\frac{Z|e^-|^2}{4\pi\epsilon_0 r^2} = \frac{m_e v^2}{r} \quad (3)$$

$$v = \sqrt{\frac{e^2}{4\pi\epsilon_0 m_e r}} \cdot Z \quad (4)$$

where we have used the shorthand notation  $|e^-| = e$ .

From (4) we can estimate the radius for the electron orbit,

$$r = \frac{e^2}{4\pi\epsilon_0 m_e v^2} Z, \quad (5)$$

which means that the radius depends on the total number of protons in the nucleus,  $Z$ , and also on the electron's velocity. Here we can make some definite estimates and see whether our estimates are reasonable, i.e., agrees or does not violate experimental data. According to the special theory of relativity, no greater velocity can any particle possess than the speed of light. More precisely, for particles with mass like electrons, we know that their velocity is limited by  $v < c$  where  $c$  is the speed of light in vacuum. Substituting for velocity  $v = c = 3 \times 10^8 \text{m/s}$ , electric charge  $e = 1.602 \times 10^{-19} \text{C}$ , mass of the electron  $m_e = 9.109 \times 10^{-31} \text{kg}$  and  $\epsilon_0 = 8.854 \times 10^{-12} \text{F/m}$  [5], we have

$$r > 2.813 \times 10^{-15} \text{m},$$

where we have taken  $Z = 1$  and didn't consider the relativistic mass. This means that we have a lower limit for the radius of an electron's orbit around the central nucleus, which is consistent with the experimental observed data where radius of electronic orbits are typically of the order  $10^{-10} \text{m}$ .

### A. Limitations of the model

Even though this model could explain some features of the atomic structure concerning the scattering data, there was nonetheless problems that could not be explained just by classical mechanics analysis. Since protons and electrons are charged particles, electromagnetic forces do play their role in this interaction, and according to Maxwell's equations, an accelerated electron emits radiation (and therefore energy), so that electrons moving around the nucleus would be emitting energy. This radiated energy would of course lead to the downspiraling of electrons around the nucleus until hitting it. Classical theoretical calculations done predicted that all electrons orbiting around a nucleus would hit it in less than a second!

However, what we observe is that there is electronic stability, and therefore the model had to be reviewed.

### III. BOHR'S HYPOTHESIS

Analysis of the hydrogen spectrum which showed that only light at certain definite frequencies and energies were emitted led Niels Bohr to postulate that the circular orbit of the electron around the nucleus is quantized, that is, that its angular momentum could only have certain discrete values, these being integer multiples of a certain basic value [6]. This was his “*ad hoc*” assumption, introduced by hand into the theory. In 1913, therefore, he proposed the following for the atomic model [7]:

1. The atom would be composed of a central nucleus where the positive charges (protons) are located;
2. Around the central nucleus revolved the electrons in equal number as the positive charges present in the nucleus. The electrons orbiting such a nucleus had discrete quantized energies, which meant that not any orbit is allowed but only certain specific ones satisfying the energy quantization requirements;
3. The allowed orbits also would have quantized or discrete values for orbital angular momentum, according to the prescription  $|\mathbf{L}| = n\hbar$  where  $\hbar = \frac{h}{2\pi}$  and  $n = 1, 2, 3, \dots$ , which meant the electron's orbit would have specific minimum radius, corresponding to the angular momentum quantum number  $n = 1$ . That would solve the problem of collapsing electrons into the nucleus.

Two colloraries for Bohr's assumptions do follow: First, from item 2. above, the laws of classical mechanics cannot describe the transition of an electron from one orbit to another, and second, when electrons do make a transition from one orbit to another, the energy difference is either supplied (transition from lower to higher energy orbits) or carried away (transition from higher to lower energy orbits) in discrete values. Only decades later (1926) G.N.Lewis coined the name *photon* for the particle carrier of this quantum of energy.

First, let us follow the usual pathway where Bohr's quantization is introduced. Using Newton's second law for the electron moving in a circular orbit around the nucleus, and thus subject to Coulomb's law, we have:

$$\frac{e^2}{4\pi\epsilon_0 r^2} = m \frac{v^2}{r}. \quad (6)$$

This allows us to calculate the kinetic energy of the electron in such an orbit:

$$E_k = \frac{1}{2}mv^2 = \frac{e^2}{8\pi\epsilon_0 r}. \quad (7)$$

The potential energy for the system proton-electron on the other hand is given by

$$E_p = -\frac{e^2}{4\pi\epsilon_0 r}, \quad (8)$$

where  $r$  is the radius of the electronic orbit.

Therefore, the total energy for the system is

$$E = -\frac{e^2}{8\pi\epsilon_0 r}. \quad (9)$$

This result would suggest that, since the radius can have any value, the same should happen with the angular momentum  $L$ .

$$L = pr \sin \theta = pr, \text{ where } \theta = 90^\circ \quad (10)$$

that is, the angular momentum depends on the radius. The linear momentum of the electron is given by

$$p = mv. \quad (11)$$

Therefore the problem of quantizing the angular momentum  $L$  reduces to the quantizing of the radius  $r$ , which depends on the total energy (9). Just here Bohr introduced an *additional hypothesis*, in that the angular momentum of the electron is quantized, i.e.,

$$L_{\text{Bohr}} = n\hbar, \quad (12)$$

where  $\hbar = \frac{h}{2\pi}$ . In this manner he was able to quantize the other physical quantities such as the total energy. This is the usual pathway wherein the textbooks normally follow in their sequence of calculations.

#### IV. ORIGIN OF ORBITAL QUANTIZATION

The total energy of the atom can be viewed from the dynamics of the atomic model constituents, i.e., electron-proton (see section 2). According to the deduction given in the Appendix, *modulus its sign*, the total energy for the system electron-proton is

$$E = \frac{mv^2}{2}. \quad (13)$$

Note that we can accommodate either the plus or minus sign here, depending on where we put the ground zero potential energy reference for the electron. Knowing that the scalar orbital velocity of the electron is

$$v = 2\pi r f, \quad (14)$$

where  $f$  is the orbital frequency and  $r$  is the radius of the electronic orbit. Substituting this  $v$  into ( 13 ) we then have:

$$E = 2\pi^2 m r^2 f^2 \quad (15)$$

so the ratio of the system's total energy variation with respect to the frequency is

$$\left. \frac{dE}{df} \right|_{\text{System}} = 4\pi^2 m r^2 f = 2\pi(2\pi r f)(mr) = 2\pi m v r = 2\pi p r = 2\pi L \quad (16)$$

Here the variation of total energy with frequency is subscripted by "System" because it comes from the classical dynamics of the system. Note too, that this variation defines the minimum for the total energy since the second derivative with respect to frequency is positive. Moreover, such variation can be rewritten in such a way as to be proportional to the angular momentum  $L$ .

Now, following Planck, let us consider that in the interaction of radiation (light) with matter, radiation 's energy is quantized according to

$$E = nh\nu \quad (17)$$

where  $n$  is a natural number ( $n = 1, 2, 3, \dots$ ),  $h$  is the Planck 's constant and  $\nu$  is the frequency of the interacting radiation. As the interaction occurs, this energy varies with respect to its frequency as:

$$\left. \frac{dE}{d\nu} \right|_{\text{Planck}} = nh. \quad (18)$$

Here we note that in (16)  $f$  is the frequency of the orbital movement of the electron, while in (18)  $\nu$  is the frequency of the radiation, and they are, of course, different in principle, so we cannot equate the two equations. It is here that Bohr 's "correspondence principle" comes into play, according to which it is hypothesized that both frequencies are the same, that is,  $\nu = f$ .

Using Bohr 's correspondence principle we can therefore write

$$\left. \frac{dE}{df} \right|_{\text{Planck}} = nh. \quad (19)$$

By energy conservation and correspondence principle it follows that

$$\left. \frac{dE}{df} \right|_{\text{System}} = \left. \frac{dE}{df} \right|_{\text{Planck}}$$

from which

$$L = \frac{nh}{2\pi}. \quad (20)$$

This is exactly Bohr 's angular momentum quantization.

## V. CONCLUSIONS

In our work here we have deduced the angular momentum quantization for atomic orbitals firstly proposed and hypothesized by Bohr by making use of three principles: energy conservation, correspondence principle and Planck's quantization for the radiation interacting with matter.

Our deduction here therefore differs from the standard one where the quantization for the angular momentum for atomic orbitals is arrived at by comparing the energy of rotating electron with half of the energy of an oscillator associated with it.

Instead of comparing the energies involved in the rotating electron with the quantized energy for the oscillator associated with it, our starting point is the energy conservation in the variation of energy of electron's movement with the variation of energy carried by the radiation interacting with the atom, which is quantized according to Planck. This leads to the angular momentum quantization.

In this work we have shown that Planck's fundamental assumption of energy quantization is more fundamental than Bohr's assumption of angular momentum quantization. In fact, we have shown that Bohr's rule for angular momentum quantization can be derived from either assuming quantization for the oscillator energy (traditional view) or by assuming that the variation of energies as the atom interacts with radiation is quantized.

## VI. APPENDIX

In this Appendix we will show that the total energy for the system electron-proton depends on the scalar orbital velocity of the electron. From classical mechanics, an electron orbiting a proton in a circular orbit obeys the following equilibrium of forces:

$$\frac{e^2}{4\pi\epsilon r^2} = \frac{mv^2}{r} \quad (21)$$

which results in

$$\frac{e^2}{4\pi\epsilon r} = mv^2 \quad (22)$$

The total energy  $E$  for the electron-proton system is equal to the sum of its kinetic energy  $E_k$  and its potential energy  $E_p$ .

$$E = E_k + E_p \quad (23)$$

where

$$E_k = \frac{mv^2}{2} \quad (24)$$

and

$$E_p = -\frac{e^2}{4\pi\epsilon r} \quad (25)$$

The negative sign for the potential energy indicates that our zero reference for it is at infinity. Substituting these in (23) we have:

$$E = \frac{mv^2}{2} - \frac{e^2}{4\pi\epsilon r} \quad (26)$$

From (22), we obtain:

$$E = \frac{mv^2}{2} - mv^2 = -\frac{mv^2}{2}$$

Therefore, the total energy  $E$  for the electron-proton system is:

$$E = -\frac{mv^2}{2}$$

Again, the negative sign here defines the bound state of the atom.

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